EXPONENTIAL LONGITUDINAL PROFILES OF STREAMS

PETER H. MORRIS* AND DAVID J. WILLIAMS
Department of Civil Engineering, The University of Queensland, Brisbane, Australia

Received 26 August 1994; Revised 13 November 1995; Accepted 5 February 1996

ABSTRACT

A theoretical model is presented which shows that streams with low solids concentrations and low lateral inflows, whose bed load sediments undergo either comminution or hydraulic sorting under steady or quasi-steady conditions, have exponential profiles. Similar streams in which both communition and sorting are significant have exponential profiles only if they are short.

Sediment threshold and flow depth estimates based on the theoretical model are consistent with field and laboratory data from the literature. A comparison of the model and communition and sorting data from the literature strongly suggests that hydraulic sorting and comminution dominate in short and long natural streams, respectively. No examples of natural streams of intermediate length with exponential longitudinal profiles were found, suggesting that neither sorting nor comminution is dominant in such streams. © 1997 by John Wiley & Sons, Ltd.

Earth surf. processes landf., **22,** 143–163 (1997)
No. of figures: 8 No. of tables: 3 No. of refs: 69
KEY WORDS comminution; exponential stream profiles; hydraulic sorting; particle size distributions

INTRODUCTION

It has long been recognized that both the longitudinal slopes of many natural streams and the particle size of their bed sediments vary exponentially or nearly so (Sternberg, 1875; Schoklitsch, 1914; Yatsu, 1955; Tanner, 1971; Graf, 1988). Similar phenomena have been observed on alluvial fans (Krumbein, 1937; Bluck, 1964) and on deltas formed by mine tailings (Williams and Morris, 1989; Morris, 1993). In the past, exponential stream profiles have been linked to Sternberg's (1875) exponential abrasion law by the simple assumption that the river slope is either proportional to, or a power function of, the size of the bed load material (Shulits, 1941; Simons, 1977). More recently, the exponential profiles of mine tailings deltas have been linked to hydraulic sorting (Morris, 1993).

In this paper, the slope and bed particle size equations are derived for natural streams with low solids concentrations, which have reached equilibrium under steady or quasi-steady hydraulic conditions, and whose bed load sediments undergo both comminution and hydraulic sorting. It is shown that exponential longitudinal stream profiles may result if the bed load sediments are subject to comminution, hydraulic sorting, or both. Departures from the model such as unsteady flow, unusual sediment particle size distributions, the confluence of streams, and external controls such as rock bars or dams lead to non-exponential stream profiles.

THEORETICAL MODEL

The theoretical model is restricted to flows with solids concentrations which are sufficiently low that the momentum of the solids is negligibly small compared to that of the water. The governing equations of mobile bed flow at low solids concentrations comprise the continuity equations for water and sediment and the momentum equation for water (de Vries, 1973; Chen *et al*., 1974; Cunge *et al*., 1980). However, since sediment transport is of primary interest in this paper, it is sufficient to consider only the momentum equation for water and continuity of the deposition of sediment on the bed.

* Correspondence to: P. H. Morris

CCC 0197-9337/97/020143–21

Momentum equation for water

The momentum equation for water is (Chen *et al*., 1974; Cunge *et al*., (1980)

$$\frac{\delta Q_w}{\delta t} + \frac{\delta(\beta Q_w^2/A)}{\delta x} + gA\frac{\delta H}{\delta x} = gA\,(S{-}S_f) + q_1 u_1 \tag{1}$$

where Q_w is the water discharge, t is the elapsed time, β is the Boussinesq velocity distribution coefficient, A is the cross-sectional area of flow, x is the longitudinal coordinate relative to an arbitrary datum, g is the acceleration due to gravity, H is the altitude of the water surface relative to an arbitrary datum, S is the bed slope, S_f is the friction slope, q_1 is the continuous lateral inflow of water per unit length of bed, and u_1 is the velocity of the lateral inflow in the direction of the main flow (Figure 1).

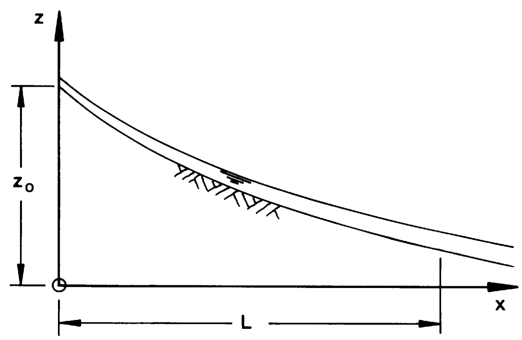

Figure 1. A diagrammatic representation of an exponential segment of a stream or alluvial fan

In many natural streams, even on steep slopes and in floods or other flows involving high accelerations (Henderson, 1966; Harder and Armacost, 1966; Yu and McNown, 1964; Iwasaki, 1967)

$$\frac{\delta Q_w}{\delta t} \ll gAS \tag{2}$$

$$\frac{\delta(BQ_w^2/A)}{\delta x} \ll gAS \tag{3}$$

and

$$gA\frac{\delta H}{\delta x} \ll gAS \tag{4}$$

Also, except in tropical areas, the momentum of the lateral inflow is usually insignificant (Cunge *et al*., 1980). That is,

$$q_1 u_1 \ll gAS \tag{5}$$

Under the constraints of Inequalities 2 to 5, Equation 1 reduces to

$$S_{\mathrm{f}} = S \tag{6}$$

Equation 6, which is consistent with data from studies of the River Rhône in France, in its natural state (Cunge *et al*., 1980), is equivalent to the assumption of quasi-steady flow. This remains valid for the study of bed profile propagation even when the flow varies rapidly in time and space, provided that the volumetric solids concentration is less than about 0·2 per cent and the flow Froude number is less than about 0·7 (de Vries, 1973; Ribberink and van der Sande, 1984; Morris and Williams, 1996). However, these low limits on the solids concentration and the flow Froude number do not apply to the slowly varying flows typical of many natural streams.

Deposition on the bed

For quasi-steady flow, the continuity equation for deposition on the bed is given by

$$-\frac{\mathrm{d}Q_{\mathrm{x}}^{B}}{\mathrm{d}x} + \eta q = \frac{\mathrm{d}V_{\mathrm{D}}}{\mathrm{d}t} \tag{7}$$

where Q_{x} is the bed load (sediment) discharge at $x = x$, $\mathrm{d}Q_{\mathrm{x}}^{\mathrm{B}}$ is that part of the derivative of Q_{x} associated with deposition on the bed, q is the lateral sediment inflow per unit length of bed, η is the fraction of q that passes to the bed, and V_{D} is the volume of sediment deposited per unit length of bed.

Equation 7 implies that the lateral sediment inflow can be treated independently of the remainder of the sediment load. This approach is reasonable if the inflow is relatively small (as required by Equation 1) and if the inflowing sediment is sufficiently fine that most of it is retained in the bed load and suspended load. The term ηq is then small compared to the others in Equation 7. (This simplified approach was adopted due to a paucity of relevant data on the relative magnitude and particle size distributions of lateral sediment inflows.)

The derivative of V_{D} with respect to time is given by

$$\frac{\mathrm{d}V_{\mathrm{D}}}{\mathrm{d}t} = (1 - \lambda)\,\frac{\mathrm{d}wz}{dt} \tag{8}$$

where λ is the bed porosity, w is the bed width, and z is the altitude of the bed relative to an arbitrary datum (Figure 1). Since λ varies slowly with particle diameter (Carling and Reader, 1982), it is assumed to be constant.

For quasi-steady conditions, ηq is constant and the derivatives with respect to t of w and z equal zero and are constant, respectively. Equation 7 then reduces to

$$-\frac{\mathrm{d}Q_{\mathrm{x}}^{\mathrm{B}}}{\mathrm{d}x} = (1 - \lambda)\, w\,\frac{\mathrm{d}z}{\mathrm{d}t} - \eta q \tag{9}$$

where $(1–\lambda)\, w\, \mathrm{d}z/\mathrm{d}t - \eta q$ is a constant.

The longitudinal stream profile can be found by expressing $\mathrm{d}Q_{\mathrm{x}}^{\mathrm{B}}/\mathrm{d}x$ as a function of x and S. This can be done by considering sediment transport, comminution and deposition, and the particle size distribution, volume per unit length of bed, and velocity of the bed load.

Sediment transport and deposition

Sediment threshold data can be expressed conveniently in terms of the dimensionless bed shear stress θ due to bed roughness (Shields, 1936; Bagnold, 1956), and the dimensionless particle size D_* (Bonnefille, 1963; van Rijn, 1984) defined by

$$\theta = \frac{h' S_{\mathrm{f}}}{(G_{\mathrm{s}} - 1)D} \tag{10}$$

and

$$D_* = D\left(\frac{(G_s - 1)g}{\nu^2}\right)^{1/3} \tag{11}$$

respectively, where h' is that part of the hydraulic radius associated with surface drag, G_s is the specific gravity of the sediment, D is the particle size, and ν is the kinematic viscosity of water. The characteristic particle size D of a poorly sorted sediment may be taken as the median particle size (Williams and Morris, 1989; Morris, 1993). However, the maximum particle size has also been used in many field studies (Tables II and III). Since the G_s of sediments in natural streams varies within a rather narrow range, it is assumed constant (Bogardi, 1974; Carling 1983; Ohmori, 1991).

Natural streams are, as a rule, very wide compared to their depths (Kamphuis, 1974; Yalin, 1992). Consequently, h, the hydraulic radius, is equal to the average depth of flow over the stream cross-section.

The depth of flow h and h' are related by

$$h = h' + h'' \tag{12}$$

where h'' is that part of the hydraulic radius associated with form drag due to ripples or dunes on the bed. Since the variation of the ratio h'/h'' with θ and D_* is uncertain (Raudkivi, 1967; Yalin, 1992), it is assumed here that it is constant over limited ranges of θ and D_*. Since, for quasi-steady flow, h is effectively constant, h' and h'' may also be taken to be constant.

Over limited ranges of D_*, the relationships between θ and D_* can be expressed as

$$\theta = aD_*^b \tag{13}$$

where a and b are dimensionless constants.

Experimental values of a and b at $\theta = \theta_t$, the dimensionless shear stress at the threshold of sediment motion, for turbulent flows with low concentrations of uniform sediments, are given in Table I. (A different threshold applies for laminar flows (Yalin and Karahan, 1979; Govers, 1987) but these are rare in natural streams and on alluvial fans.)

Table I. The range of the dimensionless particle size D_* and the coefficients a and b in Equation 13 for turbulent flow with low bed load concentrations of uniform sediments in laboratory flumes (after Morris, 1993)

D_*	a	b	Reference
<3	0·135	–0·39	adapted from Yalin and Karahan (1979)
4–10	0·14	–0·64	van Rijn (1984), adapted from Shields (1936)
10–18	0·04	–0·10	
18–150	0·013	0·29	
>150	0·055	0	

Since the value of θ_t is a function of the sediment solids concentration (Bagnold, 1956), the bed slope and the roughness of the bed relative to the depth of flow (Ashida and Bayazit, 1973; Bathurst *et al*., 1987), the particle size and specific gravity distributions (Andrews, 1983; Komar, 1987), and the particle shape (Komar and Li, 1986), as well as of D_*, values significantly different from those given in Table I occur both in nature and in the laboratory. (The effect on entrainment of seepage flow into and out of the beds of natural streams is probably minimal if mud seals do not form (Harrison and Clayton, 1970; Martin, 1970).)

Values of a and b based on flow measurements in a variety of laboratory flumes and pipes, natural streams and a tidal channel are given in Table II. The threshold lines corresponding to the data from Tables I and II are plotted in Figure 2. Baker (1973), Novak and Nalluri (1974) and Carling (1983) give Equation 13 or an equivalent equation in terms of bed shear stress, particle Reynolds number or particle size. Except for changes

of variable and conversion to metric units, these data are included unmodified in Table II. In the remaining cases, least squares regression methods (assigning all errors to θ_t) were used to evaluate a and b. (Some threshold lines with levels of significance p greater than 10 per cent were discarded.) If no specific gravity data were available, a value of 2·65 (Bogardi, 1974; Ohmori, 1991) was assumed.

All of the field data given in Table II refer to poorly sorted sediments and are based on the maximum rather than the median particle size. The θ_t estimates (Equation 10) are all based on the depth of flow h rather than h'. In the flows where pronounced bed forms occurred (White River, Lake Missoula floods, Rubicon River) or may have occurred (Oak Creek, Eel River), it is possible that h'' (Equation 12) was significant and that θ_t was consequently overestimated. The θ_t values quoted for Carl Beck (line 6 in Figure 2) are too high because bank friction was not properly accounted for (Komar and Li, 1986). The Oak Creek and the Lake Missoula lines (8 and 11, respectively) both refer to coarse boulders deposited by floods, and consequently are truncated at the low end of their ranges of D_*. the scatter of the lines plotted in Figure 2 emphasizes the importance of the prevailing hydraulic conditions in determining the threshold of sediment motion.

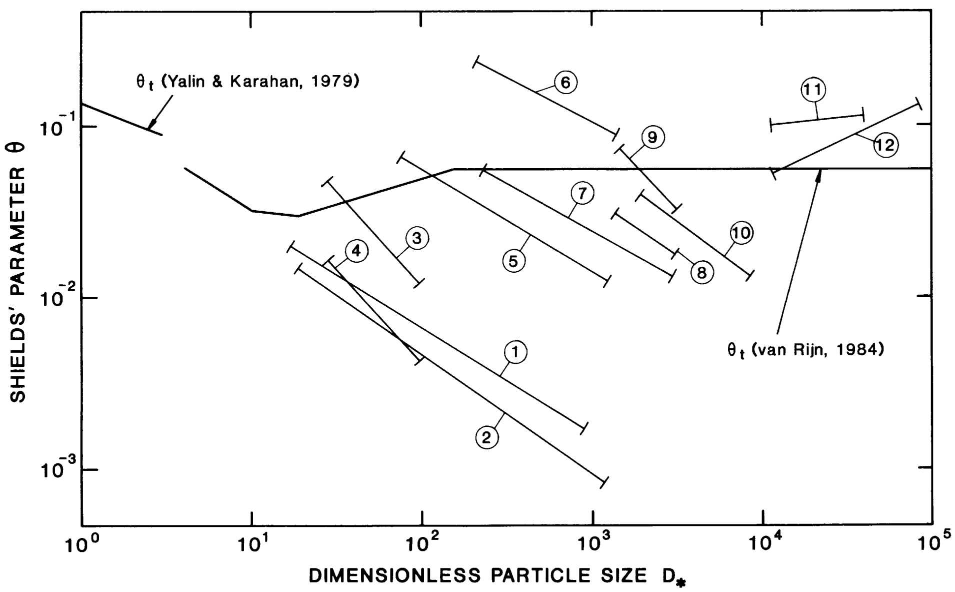

Figure 2. The Shields' θ_t required for entrainment of sediment particles versus the dimensionless particle size D_*. The lines are identified in Table II

Substituting Equations 6, 10 and 11 into Equation 13 gives

$$S = C_1 D^{1+b} \tag{14}$$

where

$$C_1 = \frac{a}{h'}\left(\frac{g}{v^2}\right)^{b/3}(G_s - 1)^{1+b/3} \tag{15}$$

Over limited ranges of D_*, C_1 is a constant.

Table II. Bed particle size data and the coefficients *a* and *b* in Equation 13 for four laboratory flumes and eight natural streams

	D_{max} (mm)	D_*	*a*	*b*	*R*	*p* (%)	Reference
Laboratory data							
1 flume	–	17–890	$1·14\times10^{-1}$	$-6·19\times10^{-1}$	–	–	Novak and Nalluri (1974)
2 pipe	–	19–1130	$1·16\times10^{-1}$	$-7·01\times10^{-1}$	–	–	Novak and Nalluri (1974)
3 flume	–	28–96	1·93	−1·11	−0·996	<0·04	Ippen and Verma (1955)
4 flume	–	28–96	$6·39\times10^{-1}$	−1·10	−0·973	<0·04	Ippen and Verma (1955)
Field data							
5 The Solent	3·0–48·2	76–1220	$8·76\times10^{-1}$	$-5·98\times10^{-1}$	−0·865	<0·04	Hammond *et al.* (1984)
6 Carl Beck	8·8–56·3	210–1370	3·78	$-5·20\times10^{-1}$	–	–	Carling (1983)
7 Great Eggleshope Beck	8·9–114	230–2930	1·14	$-5·60\times10^{-1}$	–	–	Carling (1983)
8 Oak Creek	53–117	1390–3080	3·90	$-6·72\times10^{-4}$	−0·682	<0·04	Milhous (1973); Costa (1983)
9 Middle Fork Eel River	58–122	1450–3100	$1·77\times10^{2}$	−1·07	−0·950	0·15	Ritter (1967)
10 White River	76–335	1920–8480	$1·01\times10^{1}$	$-7·33\times10^{-1}$	−0·562	2·2	Fahnstock (1963)
11 Lake Missoula floods	450–1570	11400–39700	$3·56\times10^{-2}$	$1·11\times10^{-1}$	–	–	Baker (1973)
12 Rubicon River	450–3290	11600–83200	$6·41\times10^{-4}$	$4·69\times10^{-1}$	0·703	3·3	Scott and Gravlee (1968)

For $b > -1$, Equation 14 implies that larger sediment particles require steeper slopes to be transported and will therefore be more readily deposited than smaller particles. This constitutes normal sorting and is consistent with the downstream decrease in both slope and particle size commonly observed in rivers and on alluvial fans and mine tailings deltas (Shulits, 1941; Williams and Morris, 1989). For $b < -1$, Equation 14 implies that smaller particles are more readily deposited than larger particles and hence that the particle size increases with increasing distance downstream. This constitutes reverse sorting and is associated with poorly sorted sediments (Komar, 1987). It is relatively uncommon but has been observed both in the laboratory (Everts, 1973) and in natural streams (Hack, 1957; Ritter, 1967).

Comminution of sediments

As bed load particles move downstream, their weight is reduced by abrasion and solution. This process is described by (Sternberg, 1875; Schoklitsch, 1933; Knighton, 1980)

$$W_x = W_o \exp(-\phi x) \tag{16}$$

where ϕ is a constant, W_x is the weight of the characteristic bed particle at $x = x$, and

$$W_x = \rho g \psi D_x^3 \tag{17}$$

where ρ is the density of the sediment, ψ is a shape factor, and D_x is the characteristic particle size at $x = x$. In this analysis it is assumed that ψ is constant, implying that comminution does not change the shape of the sediment particles. It is also assumed that the fines produced pass to the suspended load.

Combining Equations 16 and 17 gives

$$D_x = D_o \exp(-\alpha_1 x) \tag{18}$$

where α_1 equals $\phi/3$. Although Equation 18 is written in terms of the characteristic particle size, it applies equally to all bed load particles.

Bogardi (1974) gives an 'average' α_1 value of $3{\cdot}3 \times 10^{-6}$ m^{-1} for all rocks, while Adams (1979) gives α_1 values of $2{\cdot}9 \times 10^{-6}$ m^{-1} and $9{\cdot}7 \times 10^{-6}$ m^{-1} for quartz and chlorite schist, a relatively weak rock, respectively.

Particle size distributions of sediments

For sediments with constant G_s, the volumetric and gravimetric particle size distributions are indistinguishable. The gravimetric size distributions of natural sediments often approximate to either (truncated) log-normal distributions (Krumbein, 1938; Blatt *et al.*, 1980) or Rosin-Rammler distributions (Kittleman, 1964; Ibbeken, 1983; Shih and Komar, 1990).

Morris (1993) observed that log-normal distributions which encompass a relatively large range of particle sizes approximate to log-uniform distributions, which are described by

$$C(V) = \frac{\mathrm{Ln}(D) - \ln(D^{\min})}{\ln(D^{\max}) - \ln(D^{\min})} \tag{19}$$

where $C(V)$ is the fractional volume comprising particle sizes less than D, and $D^{\max}$ and $D^{\min}$ are constants (Tarantola, 1987).

The Rosin-Rammler distribution is given by

$$C(V) = 1 - \exp[-(D/\bar{D})^N] \tag{20}$$

where $\bar{D}$ is the absolute size constant, which corresponds to the 63·2th percentile of the cumulative distribution, and N is the distribution constant (Bennet, 1937). The value of N may range from 0·6 to infinity, but, in practice,

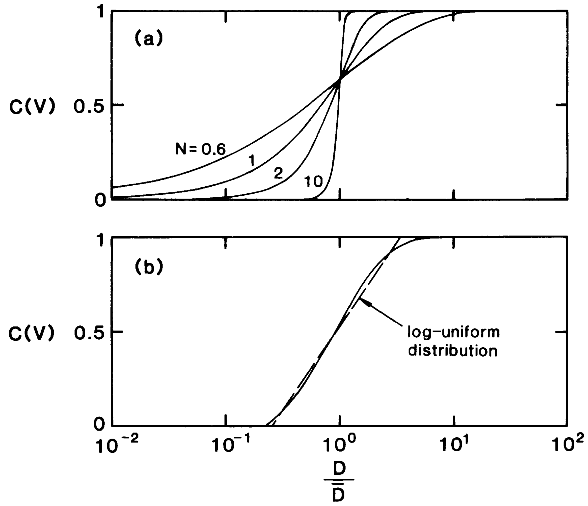

Figure 3. (a) Rosin-Rammler cumulative distributions (Equation 20) with distribution constants N in the range of 0·6 to 10. (b) A log-uniform cumulative distribution (Equation 19) superimposed on the Rosin-Rammler distribution with $N = 1$, truncated at the 20th percentile

approximates to unity (Bennett, 1937). Figure 3 shows Rosin-Rammler distributions with N values in the range of 0·6 to 10, and a log-uniform distribution superimposed on a truncated Rosin-Rammler distribution with N equal to unity. The close match shows that sediments with Rosin-Rammler distributions truncated by the division of the sediment into suspended load and bed load are also described by Equation 19 to a good approximation.

Volume of bed load per unit length of bed

At $x = 0$, the particle size distribution of the bed load is given by (Equation 19)

$$C(V) = \frac{\mathrm{Ln}(D) - \mathrm{Ln}(D_0^{\min})}{\mathrm{Ln}(D_0^{\max}) - \mathrm{Ln}(D_0^{\min})} \tag{21}$$

In normal sorting, the largest bed load particles are deposited on the bed. That is, D_x equals $D_x^{\max}$. Notionally reversing the effects of communution gives the equivalent bed load size at $x = 0$ (Equation 18)

$$D_x^{\max\prime} = D_x \exp(\alpha_1 x) \tag{22}$$

Similarly

$$D_x^{\min\prime} = D_x^{\min} \exp(\alpha_1 x) \tag{23}$$

Also, Equations 17 and 18 imply that, for comminution only,

$$V_x = V_0 \exp(-\phi x) \tag{24}$$

where V_x is the volume of bed load per unit length of bed (exclusive of lateral sediment inflows). Hence,

$$V_x' = V_x \exp(\phi x) \tag{25}$$

Combining Equations 21 to 23 and 25 leads to

$$V_x = V_0 \exp(-\phi x)\left[\frac{\ln(D_x) - \ln(D_x^{\min})}{\ln(D_0^{\max}) - \ln(D_0^{\min})}\right] \tag{26}$$

and

$$\frac{\mathrm{d}V_x}{\mathrm{d}x} = -\phi V_x + V_0 \exp(-\phi x)\left[\frac{D_x^{-1}\dfrac{\mathrm{d}D_x}{\mathrm{d}x} - D_x^{\min^{-1}}\dfrac{\mathrm{d}D_x^{\min}}{\mathrm{d}x}}{\ln(D_0^{\max}) - \ln(D_0^{\min})}\right] \tag{27}$$

However (excluding lateral sediment inflows)

$$\frac{\mathrm{d}V_x}{\mathrm{d}x} = \frac{\mathrm{d}V_x^B}{\mathrm{d}x} + \frac{\mathrm{d}V_x^S}{\mathrm{d}x} \tag{28}$$

where $\mathrm{d}V_x^B$ and $\mathrm{d}V_x^S$ are associated with deposition on the bed and transfer to the suspended load, respectively.

Since $\mathrm{d}V_x^S$ arises only from comminution, Equation 24 implies that

$$\frac{\mathrm{d}V_x^S}{\mathrm{d}x} = -\phi V_x \tag{29}$$

Combining Equations 27 to 29 then gives, for normal sorting with comminution,

$$\frac{\mathrm{d}V_x^B}{\mathrm{d}x} = V_0 \exp(-\phi x)\left[\frac{D_x^{-1}\dfrac{\mathrm{d}D_x}{\mathrm{d}x} - D_x^{\min-1}\dfrac{\mathrm{d}D_x^{\min}}{\mathrm{d}x}}{\ln(D_0^{\max}) - \ln(D_0^{\min})}\right] \tag{30}$$

In reverse sorting, bed load particles of size $D_x^{\min}$ may be transferred to the suspended load as well as deposited on the bed. On the basis of similarity, the fraction ζ ($0 \le \zeta \le 1$) of particles size $D_x^{\min}$ deposited on the bed in quasi-steady flow is constant for all x.

Proceeding as for normal sorting and applying Equation 18 then leads to

$$\frac{\mathrm{d}V_x^B}{\mathrm{d}x} = -\varsigma V_0 \exp(-\phi x)\left[\frac{D_x^{-1}\dfrac{\mathrm{d}D_x}{\mathrm{d}x} + \alpha_1}{\ln(D_0^{\max}) - \ln(D_0^{\min})}\right] \tag{31}$$

Mean velocity of bed load

The mean velocity of the bed load is given by

$$\bar{U}_x = \frac{1}{V_x}\int_{D_x^{\min}}^{D_x^{\max}} U_D \,\mathrm{d}V_x \tag{32}$$

where U_D is the velocity of sediment particles of size D.

If comminution is governed by Equation 18 and the fines produced pass to the suspended load, bed load particle size distributions which are initially log-uniform remain log-uniform, for both normal and reverse sorting. That is, Equation 19 applies for all x, whence

$$\mathrm{d}V_{\mathrm{x}} = \frac{V_{\mathrm{x}} D^{-1} \mathrm{d}D}{\ln(D_{\mathrm{x}}^{\max}) - \ln(D_{\mathrm{x}}^{\min})} \tag{33}$$

For deterministic sorting of uniform sediment (Kalinski, 1947)

$$U_{\mathrm{D}} = u - u_{\mathrm{D}} \tag{34}$$

where u is the velocity of flow, and u_{D} is the flow velocity at the threshold of movement for particles of size D.

For normal sorting (Kalinski, 1947)

$$u_{\mathrm{D}} = C_2 D^{1/2} \tag{35}$$

where C_2 is a constant.

Also, since particles of size $D_0^{\max}$ are deposited at $x = 0$, $U_{\mathrm{D}_0}^{\max}$ equals zero. Hence, by substitution in Equation 35, u_0 equals $u_{\mathrm{D}_0}^{\max}$, and (Equation 35)

$$C_2 = u_0 D_0^{\max -1/2} \tag{36}$$

Combining Equations 35 and 36 gives

$$u_{\mathrm{D}} = u_0 \left(\frac{D}{D_0^{\max}} \right)^{1/2} \tag{37}$$

and since, for quasi-steady flow, u effectively equals u_0, a constant, Equation 34 becomes

$$u_{\mathrm{D}} = u_0 \left[1 - \left(\frac{D}{D_0^{\max}} \right)^{1/2} \right] \tag{38}$$

Similarly, for reverse sorting,

$$u_{\mathrm{D}} = C_2 D^{-1/2} \tag{39}$$

whence

$$U_{\mathrm{D}} = u_0 \left[1 - \left(\frac{D_0^{\min}}{D} \right)^{1/2} \right] \tag{40}$$

For normal sorting and quasi-steady flow, Equations 32, 33 and 38 give

$$\overline{U}_{\mathrm{x}} = u_0 \left[1 - \frac{2 D_0^{\max^{-1/2}} \left(D_{\mathrm{x}}^{\max^{1/2}} - D_{\mathrm{x}}^{\min^{1/2}} \right)}{\ln(D_{\mathrm{x}}^{\max}) - \ln(D_{\mathrm{x}}^{\min})} \right] \tag{41}$$

and

$$\frac{\bar{U}_x}{\bar{U}_0} = \frac{D_0^{\max^{1/2}} - \dfrac{2\left(D_x^{\max^{1/2}} - D_x^{\min^{1/2}}\right)}{\ln(D_x^{\max}) - \ln(D_x^{\min})}}{D_0^{\max^{1/2}} - \dfrac{2\left(D_0^{\max^{1/2}} - D_0^{\min^{1/2}}\right)}{\ln(D_0^{\max}) - \ln(D_0^{\min})}} \tag{42}$$

where $\bar{U}_0$ is a constant, $D_x^{\max}$ equals $D_0^{\max}$ exp $(-\alpha_1 x)$, and, if bed load particles of size $D_x^{\min}$ are comminuted in accordance with Equation 18, $D_x^{\min}$ equals $D_0^{\min}$ $\exp(-\alpha_1 x)$.

Equation 42 is then indeterminate at $D_x = D_x^{\min}$, where $\bar{U}_x/\bar{U}_0$ assumes its maximum value. However, applying L'Hospital's rule leads to

$$\left(\frac{\bar{U}_x}{\bar{U}_0}\right)_{\max} = \frac{\ln\left(D_0^{\max} D_0^{\min^{-1}}\right)\left[1 - \exp(-\alpha_1 x/2) D_0^{\min^{1/2}} D_0^{\max^{-1/2}}\right]}{\ln\left(D_0^{\max} D_0^{\min^{-1}}\right) - 2\left(1 - D_0^{\min^{1/2}} D_0^{\max^{-1/2}}\right)} \tag{43}$$

A similar result is obtained if bed load particles of size $D_0^{\min}$ pass directly to the suspended load. However, the maximum of $\bar{U}_x/\bar{U}_0$ is then always less than that obtained from Equation 43.

For reverse sorting Equations 32, 33 and 40 lead to

$$\left(\frac{\bar{U}_x}{\bar{U}_0}\right)_{\max} = \frac{\ln\left(D_0^{\max} D_0^{\min^{-1}}\right)\left[1 - \exp(\alpha_1 x/2) D_0^{\min^{1/2}} D_0^{\max^{-1/2}}\right]}{\ln\left(D_0^{\max} D_0^{\min^{-1}}\right) - 2\left(1 - D_0^{\min^{1/2}} D_0^{\max^{-1/2}}\right)} \tag{44}$$

Equations 43 and 44 are identical if α_1 equals zero (no comminution). For all other α_1, the maximum of $\bar{U}_x/\bar{U}_0$ obtained from Equation 44 is less than that from Equation 43.

Figure 4 shows the variation of the range of $\bar{U}_x/\bar{U}_0$ with $D_0^{\max}/D_0^{\min}$. The upper bound of $\bar{U}_x/\bar{U}_0$ corresponds to Equation 43 with $\alpha_1 x$ taken as 3. (The maximum value of αx, which incorporates $\alpha_1 x$, for the natural streams listed in Table III is 2·1.) The minimum value for $D_0^{\max}/D_0^{\min}$ of 10 is compatible with the 'relatively large' range of particle sizes implied by Equation 19. For $D_0^{\max}/D_0^{\min}$ greater than 10, the minimum of $\bar{U}_x/\bar{U}_0$ equals unity (at $x = 0$) for all cases.

Figure 4 thus shows that the value of $\bar{U}_x/\bar{U}_0$ for natural streams ranges from 1 to a maximum of less than 3. This range is relatively small and therefore, to a good approximation, for quasi-steady flow,

$$\bar{U}_x = C_* \bar{U}_0 \tag{45}$$

where C_* is an arbitrary constant between 1 and 3. Equation 45 is valid for all bed load particles for all x.

Evaluation of $dQ_x^B/\mathrm{d}x$

The bed load discharge (excluding lateral sediment inflows) is given by

$$Q_x = \bar{U}_x V_x \tag{46}$$

Hence (Equation 45)

$$\frac{\mathrm{d}Q_x}{\mathrm{d}x} = C_* \bar{U}_0 \frac{\mathrm{d}V_x}{\mathrm{d}x} \tag{47}$$

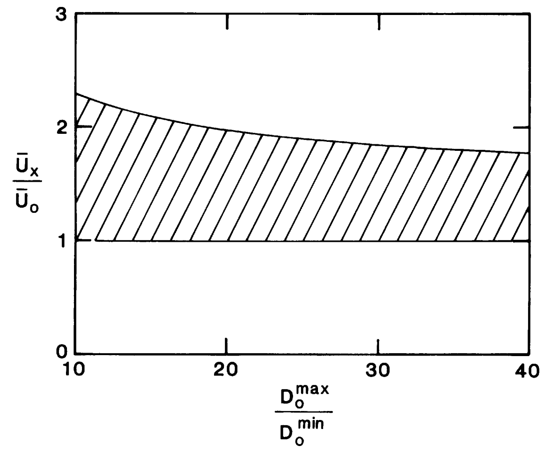

Figure 4. The variation of the range of $\bar{U}_x/\bar{U}_0$ with D_0^{max}/D_0^{min}, based on Equations 42 to 44

and, since Equation 45 is valid for all bed load particles,

$$\frac{dQ_x^B}{dx} = C_* \bar{U}_0 \frac{dV_x^B}{dx} \tag{48}$$

Also, differentiating Equation 14 and eliminating C_1 gives

$$D_x^{-1} \frac{dD_x}{dx} = (1+b)^{-1} S_x^{-1} \frac{dS_x}{dx} \tag{49}$$

For normal sorting, Equations 30, 48 and 49 give

$$\frac{dQ_x^B}{dx} = \frac{C_* \bar{U}_0 V_0 \exp(-3\alpha_1 x)}{(1+b)} \left[\frac{S_x^{-1} \frac{dS_x}{dx} - (1+b) D_x^{\min^{-1}} \frac{dD_x^{\min}}{dx}}{\ln(D_0^{\max}) - \ln(D_0^{\min})} \right] \tag{50}$$

If particles of size D_x^{min} are comminuted in accordance with Equation 18, Equation 50 becomes

$$\frac{dQ_x^B}{dx} = \frac{C_* \bar{U}_0 \zeta V_0 \exp(-3\alpha_1 x)}{(1+b)} \left[\frac{S_x^{-1} \frac{dS_x}{dx} + (1+b)\alpha_1}{\ln(D_0^{\max}) - \ln(D_0^{\min})} \right] \tag{51}$$

If bed load particles of size D_x^{min} are transferred directly to the suspended load, D_x^{min} equals D_0^{min}, a constant, and

$$\frac{dQ_x^B}{dx} = \frac{C_* \bar{U}_0 V_0 \exp(-3\alpha_1 x)}{(1+b)} \left[\frac{S_x^{-1} \frac{dS_x}{dx}}{\ln(D_0^{\max}) - \ln(D_0^{\min})} \right] \tag{52}$$

For reverse sorting, Equations 31, 48 and 49 give

$$\frac{\mathrm{dQ}_{\mathrm{x}}^{\mathrm{B}}}{\mathrm{d}x} = \frac{C_{*}\overline{U}_{0}\zeta V_{0}\exp(-3\alpha_{1}x)}{(1+b)}\left[\frac{S_{x}^{-1}\dfrac{\mathrm{d}S_{x}}{\mathrm{d}x}+(1+b)\alpha_{1}}{\ln(D_{0}^{\max})-\ln(D_{0}^{\min})}\right] \tag{53}$$

Longitudinal profile and sorting equations

Since Equations 51 to 53 express $\mathrm{d}Q_{\mathrm{x}}^{\mathrm{B}}/\mathrm{d}x$ as a function of x and z only, Equation 9 can now be solved. Combining Equations 9 and 51, for normal sorting with variable $D_{\mathrm{x}}^{\min}$, gives

$$\frac{\mathrm{d}S_{\mathrm{x}}}{\mathrm{d}x} + (1+\mathrm{b})\left[\alpha_{1} + \alpha_{2}\exp(3\alpha_{1}x)\right]S_{\mathrm{x}} = 0 \tag{54}$$

where α_2, a constant, is given by

$$\alpha_{2} = \frac{1}{C_{*}\overline{U}_{0}V_{0}}\left[(1-\lambda w\frac{\mathrm{d}z}{\mathrm{d}t}-\eta q\right]\left[\ln(D_{0}^{\max})-\ln(D_{0}^{\min})\right] \tag{55}$$

Solving Equation 54 and applying the boundary condition $x = 0$ at $S_{\mathrm{x}} = S_0$ gives

$$S_{\mathrm{x}} = S_{0}\exp\left[(1+b)\left\{\frac{\alpha_{2}}{3\alpha_{1}\zeta}\left[1-\exp(3\alpha_{1}x)\right]-\alpha_{1}x\right\}\right] \tag{56}$$

Combining Equations 56 and 14 gives

$$D_{\mathrm{x}} = D_{0}\exp\left\{\frac{\alpha_{2}}{3\alpha_{1}}\left[1-\exp(3\alpha_{1}x)\right]-\alpha_{1}x\right\} \tag{57}$$

Similarly, for normal sorting with constant $D_{\mathrm{x}}^{\min}$ (Equation 52),

$$S_{\mathrm{x}} = S_{0}\exp\left\{(1+b)\frac{\alpha_{2}}{3\alpha_{1}}\left[1-\exp(3\alpha_{1}x)\right]\right\} \tag{58}$$

$$D_{\mathrm{x}} = D_{0}\exp\left\{\frac{\alpha_{2}}{3\alpha_{1}}\left[1-\exp(3\alpha_{1}x)\right]\right\} \tag{59}$$

and, for reverse sorting (Equation 53),

$$S_{\mathrm{x}} = S_{0}\exp\left[(1+b)\left\{\frac{\alpha_{2}}{3\alpha_{1}\zeta}\left[1-\exp(3\alpha_{1}x)\right]-\alpha_{1}x\right\}\right] \tag{60}$$

$$D_{\mathrm{x}} = D_{0}\exp(1+b)\left\{\frac{\alpha_{2}}{3\alpha_{1}\zeta}\left[1-\exp(3\alpha_{1}x)\right]-\alpha_{1}x\right\} \tag{61}$$

The significance of α_2 becomes clear when the special case of no comminution ($\alpha_1 = 0$) is considered.

L'Hospital's rule gives

$$\lim_{\alpha_1 \to 0}\left[\frac{1 - \exp(3\alpha_1 x)}{3\alpha_1}\right] = -x \tag{62}$$

and Equations 56 to 61 then reduce to

$$S_x = S_0 \exp(-\varepsilon x) \tag{63}$$

$$D_x = D_0 \exp(-\alpha_2 x) \tag{64}$$

where ε is a constant, and

$$\alpha_2 = \frac{\varepsilon}{1+b} \tag{65}$$

For both normal sorting with variable D_x^{min} and reverse sorting, the complementary special case ($\alpha_2 = 0$) gives Equations 63 and 18, where

$$\alpha_1 = \frac{\varepsilon}{1+b} \tag{66}$$

Since α_1 cannot be negative, Equation 18 represents normal sorting.

For normal sorting with constant D_x^{min}, a straight (constant slope) profile and a constant bed particle size are obtained. A straight profile is also obtained in all cases if both α_1 and α_2 equal zero.

These results show that exponential longitudinal stream profiles may arise when either hydraulic sorting (α_2) or comminution (α_1) is dominant. However, comparison of Equations 63 and 56 (normal sorting with variable D_x^{min}) or 60 (reverse sorting), shows that exponential profiles (Equation 63) also arise if

$$\exp(3\alpha_1 x) = 1 - \frac{3\alpha_1}{\alpha_2}\left[\alpha_1 - \frac{\varepsilon}{(1+b)}\right]x \tag{67}$$

Equation 67 corresponds to the Maclaurin expansion for $\exp(3\alpha_1 x)$ truncated at the second term (Sokolnikoff and Redheffer, 1958; Thomas, 1968), provided that

$$-3\alpha_1 = \frac{3\alpha_1}{\alpha_2}\left[\alpha_1 - \frac{\varepsilon}{(1+b)}\right] \tag{68}$$

whence

$$\alpha = \frac{\varepsilon}{(1+b)} \tag{69}$$

$$D_x = D_0 \exp(-\alpha x) \tag{70}$$

where

$$\alpha = \alpha_1 + \alpha_2 \tag{71}$$

as suggested by Knighton (1980). However, Equation 58 (normal sorting with constant D_x^{min}) again leads to Equations 64 and 65.

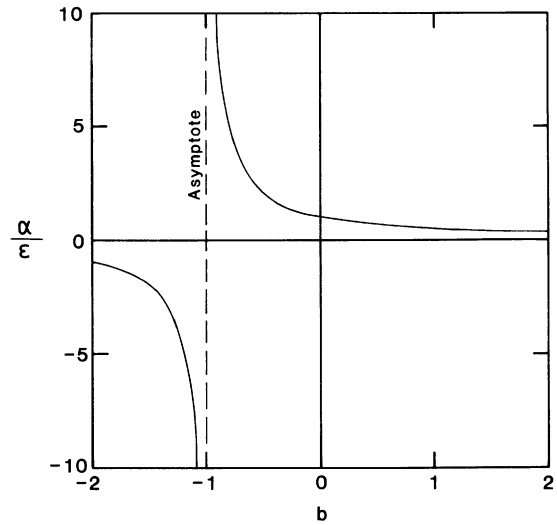

Figure 5. The variation of *b* with α/ε based on Equation 69

Since the solution of Equation 67 is an approximation, it is valid for a limited range of x. For Bogardi's (1974) average α_1 value of $3{\cdot}3\times10^{-6}$ m^{-1}, the fractional error E in $\exp(3\alpha_1 x)$, which is given by

$$E = 1 - \frac{1+3\alpha_1 x}{\exp(3\alpha_1 x)} \qquad (72)$$

reaches 0·01 for x = 14·9 km and 0·1 for x = 53·2 km. Hence, if neither comminution (α_1) nor hydraulic sorting (α_2) is dominant, exponential profiles can occur only in relatively short streams.

The variation of α/ε with b (Equation 69) is shown in Figure 5. The discontinuity at $b = -1$ corresponds to the transition from reverse to normal sorting. (This transition may occur at other values of b if the specific gravity of the bed sediment varies with distance along the stream.) For $b = -1$, Equation 14 implies a constant bed slope, but the size of the particles on the bed is undefined.

For the special cases leading to Equation 63, the datum for z can be chosen arbitrarily so that $z = 0$ at $x = \infty$, and integration gives

$$z = z_0 \exp(-\varepsilon x) \qquad (73)$$

where $z_0 = z$ at $x = 0$ (Figure 1).

Equation 73 implies that exponential longitudinal stream profiles produced by quasi-steady flows are invariant over time. If there is no deposition ($\alpha_2 = 0$) the altitude of the profile is constant. Otherwise, since dz/dt is constant (Equation 55), the altitude of the whole profile increases (or decreases) uniformly with increasing time.

Changing the datum for z in Equation 73 to $z = 0$ at $x = L$, where L is the length of an exponential river reach or delta segment, gives the dimensionless form (Figure 6)

$$\frac{z}{z_0} = A\exp\left(-\omega\frac{x}{L}\right) - A\exp(-\omega) \qquad (74)$$

where

$$\omega = \varepsilon L \qquad (75)$$

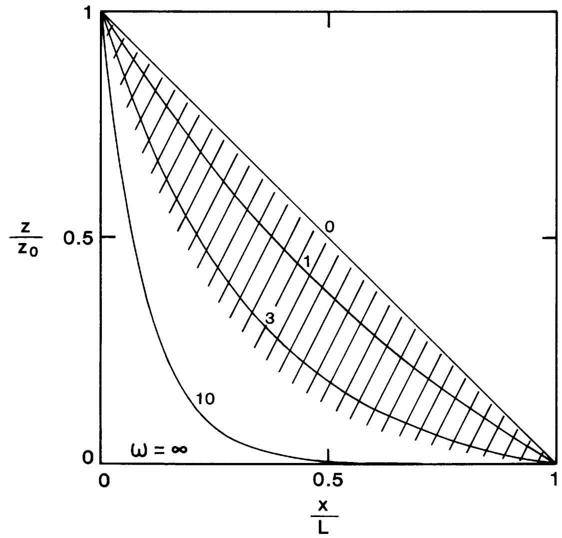

Figure 6. Dimensionless profiles of exponential stream segments based on Equation 74. The hatched area corresponds to the range of ω of from 0·082 to 3·77 for the stream segments listed in Table III

is a dimensionless constant and $A = [1–\exp(–\omega)]^{-1}$ (Williams and Morris, 1989; Morris, 1993).

Equation 70 implies that exponential stream segments end ($x = L$) when their bed loads are reduced to particles of a single size and hence are exhausted. For comminution only ($\alpha_2 = 0$), this occurs at $x = \infty$. For normal sorting with no comminution ($\alpha_1 = 0$) and negligible lateral sediment inflow to the bed load, it occurs when D_x equals D_0^{min}. Since D_0 then equals D_0^{max}, substituting for D_x, D_0 and x in Equation 64, eliminating α_2 using Equation 65, and applying Equation 75 leads to

$$\omega = (1 + b) \ln (D_0^{max}/D_0^{min}) \tag{76}$$

That is, ω is independent of L, as suggested by Williams and Morris (1989). The same result is obtained for reverse sorting with no comminution since α_2 is then negative.

COMPARISON OF MODEL AND DATA

Data from 30 natural stream segments and one alluvial fan with exponential longitudinal profiles, mostly from North America and Japan, are listed in Table III in order of decreasing length. In five cases, data from successive gravel and sand bed segments of the same stream are presented.

Equations 73, 63 and 70, or equivalent equations, are given in Yatsu (1955), Rafay (1964), Bluck (1964) and Bradley *et al*. (1972). Except for changes of variable or conversion to metric units, these data are included unmodified in Table III. In the remaining cases, particle size and altitude or slope data only are given. Values of ε and α were obtained by fitting Equations 73 or 63 and 70, respectively, to these data using least squares regression methods (Figure 7). (The longitudinal coordinate x was assumed to be error-free in all cases.) No specific gravity data were available and a constant value of 2·65 (Bogardi, 1974; Ohmori, 1991) was assumed in all cases. Stream segments with levels of significance p for either ε or α of greater than 10 per cent were discarded from the data set. Examples of segments with statistically significant exponential profiles were frequently found whose sediments did not conform (statistically) to Equation 70. In addition to deviations from the theoretical model, this reflects the relative difficulty of obtaining representative sediment samples.

The stream segments listed in Table III can be divided into two groups comprising short (<80 km) and long (>240 km) segments. The long (1 to 8) and short (9 to 31) segments have α ranging from 0·26 to 1·5 and from 5·2

Table III. Profile and bed particle size data and the coefficients a and b in Equation 13 for 30 natural stream segments and an alluvial fan

	Stream	Length (km)	D_{50} (mm)	D_*	S_O	ϵ (m^{-1})	ω	R	p (%)	D_O (mm)	α (m^{-1})	R	p (%)	a	b	Reference
1	Mississippi River	1770·0	0·12–0·72	4·0–18·2	–	$1·01\times10^{-6}$	1·78	–	–	$7·2\times10^{-1}$	$8·51\times10^{-7}$	–	–	–	$1·82\times10^{-1}$	Rafay (1964); Simons (1977)
2	Peace River	469·9	15–50	345–902	$4·59\times10^{-4}$	$1·25\times10^{-6}$	0·587	−0·999	<0·04	$3·57\times10^{-1}$	$2·04\times10^{-6}$	−0·648	8·5	–	$-3·89\times10^{-1}$	Kellerhalls *et al.* (1972)
3	Peace River	410·1	0·14–0·35	4·4–8·3	$2·05\times10^{-4}$	$3·78\times10^{-6}$	1·55	−0·999	<0·04	$3·26\times10^{-1}$	$1·56\times10^{-6}$	−0·828	4·1	–	1·43	Kellerhalls *et al.* (1972)
4	North Saskatchewan River	393·0	7–28	228–550	$5·20\times10^{-4}$	$1·42\times10^{-6}$	0·559	−0·999	<0·04	$2·17\times10^{1}$	$2·24\times10^{-6}$	−0·723	<0·04	–	$-3·66\times10^{-1}$	Kellerhalls *et al.* (1972)
5	Oldman River	282·4	16–39	354–836	$1·42\times10^{-3}$	$3·01\times10^{-6}$	0·851	−0·999	<0·04	$3·30\times10^{1}$	$3·05\times10^{-6}$	−0·919	2·5	–	$-1·06\times10^{-2}$	Kellerhalls *et al.* (1972)
6	Red Deer River	266·2	16–112	683–1520	$2·07\times10^{-3}$	$4·81\times10^{-6}$	1·28	−0·998	<0·04	$6·01\times10^{1}$	$3·01\times10^{-6}$	−0·607	1·9	–	$6·00\times10^{-1}$	Kellerhalls *et al.* (1972)
7	North Saskatchewan River	258·6	21–130	678–2430	$2·43\times10^{-3}$	$2·18\times10^{-6}$	0·563	−0·999	<0·04	$9·62\times10^{1}$	$4·94\times10^{-6}$	−0·779	1·9	–	$-5·59\times10^{-1}$	Kellerhalls *et al.* (1972)
8	Rio Grande	241·4	0·14–0·50	5·2–12·6	–	$5·72\times10^{-6}$	1·38	–	–	$5·0\times10^{-1}$	$3·67\times10^{-6}$	–	–	–	$5·59\times10^{-1}$	Rafay (1964); Simons (1977)
9	Rubicon River	78·7	457–3290*	11000–58100	$2·31\times10^{-2}$	$2·96\times10^{-5}$	2·33	−0·948	<0·04	$2·30\times10^{3}$	$2·11\times10^{-5}$	−0·898	<0·04	$1·31\times10^{-3}$	$4·02\times10^{-1}$	Scott and Gravlee (1968)
10	Kinu River	52·0	20–70	475–1770	$6·73\times10^{-3}$	$2·75\times10^{-5}$	1·43	–	–	7×10^{1}	$2·53\times10^{-5}$	–	–	$3·22\times10^{-2}$	$8·70\times10^{-2}$	Yatsu (1955); Simons (1977)
11	Standing Stone Creek	43·5	11·5–117	462–2390	$2·04\times10^{-2}$	$8·25\times10^{-5}$	3·58	−0·940	<0·04	$9·44\times10^{1}$	$3·78\times10^{-5}$	−0·802	0·4	$3·94\times10^{-5}$	1·18	Brush (1961)
12	Kinu River	40·0	0·4–0·9	8·8–22·8	$7·02\times10^{-4}$	$2·39\times10^{-5}$	0·956	–	–	9×10^{-1}	$2·38\times10^{-5}$	–	–	$3·45\times10^{-2}$	$4·2\times10^{-3}$	Yatsu (1955); Simons (1977)
13	Nagara River	36·0	0·7–1·2	16·3–30·4	$1·85\times10^{-4}$	$2·81\times10^{-5}$	1·01	–	–	1·2	$1·73\times10^{-5}$	–	–	$5·30\times10^{-3}$	$6·24\times10^{-1}$	Yatsu (1955); Simons (1977)
14	Yahagi River	35·0	1–2	21·3–50·6	$1·18\times10^{-2}$	$2·43\times10^{-5}$	0·851	–	–	2·0	$2·47\times10^{-5}$	–	–	$3·32\times10^{-2}$	$-1·62\times10^{-2}$	Yatsu (1955); Simons (1977)
15	Honey Creek	28·2	31–91	888–2040	$2·44\times10^{-2}$	$7·81\times10^{-5}$	2·20	−0·870	0·3	$8·06\times10^{1}$	$2·95\times10^{-5}$	−0·775	2·1	$7·60\times10^{-7}$	1·65	Brush (1961)
16	Knik River	25·7	40·9–327*	1030–8270	$2·33\times10^{-3}$	$4·97\times10^{-5}$	1·28	–	–	$3·27\times10^{2}$	$8·08\times10^{-5}$	–	–	$7·94\times10^{-1}$	$-3·85\times10^{-1}$	Bradley *et al.* (1972)
17	Abe River	23·0	15–90	440–2280	$1·10\times10^{-2}$	$4·3\times10^{-5}$	0·989	–	–	9×10^{1}	$7·15\times10^{-5}$	–	–	$6·22\times10^{-1}$	$-3·99\times10^{-1}$	Yatsu (1955); Simons (1977)
18	Watrase River	21·0	30–80	664–2020	$1·01\times10^{-2}$	$1·06\times10^{-4}$	2·23	–	–	8×10^{1}	$5·31\times10^{-5}$	–	–	$8·49\times10^{-5}$	$9·96\times10^{-1}$	Yatsu (1955); Simons (1977)
19	Shaver Creek	20·6	23–141	594–2770	$3·18\times10^{-2}$	$1·83\times10^{-4}$	3·77	−0·958	<0·04	$1·10\times10^{2}$	$7·49\times10^{-5}$	−0·855	<0·04	$5·37\times10^{-6}$	1·45	Brush (1961)
20	Sho River	20·0	20–50	711–1260	$7·97\times10^{-3}$	$7·13\times10^{-5}$	1·43	–	–	5×10^{1}	$2·88\times10^{-5}$	–	–	$3·40\times10^{-6}$	1·48	Yatsu (1955); Simons (1977)
21	Gillis Falls	17·7	7·1–80	348–2330	$1·37\times10^{-2}$	$1·10\times10^{-4}$	1·95	−0·928	<0·04	$9·64\times10^{1}$	$-1·08\times10^{-4}$	−0·824	0·4	$3·68\times10^{-5}$	−2·03	Hack (1957)
22	Kiso River	15·0	30–70	1050–1770	$2·49\times10^{-3}$	$4·9\times10^{-5}$	0·735	–	–	7×10^{1}	$3·48\times10^{-5}$	–	–	$3·22\times10^{-3}$	$4·08\times10^{-1}$	Yatsu (1955); Simons (1977)
23	North River	14·2	10–55	252–1450	$9·75\times10^{-3}$	$8·59\times10^{-5}$	1·22	−0·996	<0·04	$5·73\times10^{1}$	$1·23\times10^{-4}$	−0·906	0·9	$2·95\times10^{-1}$	$-3·04\times10^{-1}$	Hack (1957)
24	Watrase River	14·0	0·3–0·9	12·7–22·8	$6·17\times10^{-4}$	$2·95\times10^{-5}$	0·413	–	–	9×10^{-1}	$4·16\times10^{-5}$	–	–	$6·50\times10^{-2}$	$-2·91\times10^{-1}$	Yatsu (1955); Simons (1977)
25	Tenryu River	14·0	15–50	601–1260	$1·72\times10^{-3}$	$5·88\times10^{-6}$	0·082	–	–	5×10^{1}	$5·32\times10^{-5}$	–	–	$1·63\times10^{1}$	$-8·89\times10^{-1}$	Yatsu (1955); Simons (1977)
26	Weiker Run	13·4	95–142	2620–3520	$3·60\times10^{-2}$	$5·01\times10^{-5}$	0·669	−0·929	0·1	$1·39\times10^{2}$	$2·21\times10^{-5}$	−0·707	7·8	$2·50\times10^{-6}$	1·27	Brush (1961)
27	Nagara River	13·0	25–40	567–1010	$1·35\times10^{-3}$	$8·14\times10^{-5}$	1·06	–	–	4×10^{1}	$4·46\times10^{-5}$	–	–	$2·94\times10^{-4}$	$8·25\times10^{-1}$	Yatsu (1955); Simons (1977)
28	Tye River	10·6	230–630	6640–15400	$8·04\times10^{-2}$	$6·48\times10^{-5}$	0·688	−0·861	6·7	$6·10\times10^{2}$	$8·38\times10^{-5}$	−0·888	4·6	$4·02\times10^{-1}$	$-2·27\times10^{-1}$	Hack (1957)
29	Arrow River Canyon alluvial fan	5·5	434–1133*	7930–28700	$3·04\times10^{-2}$	$3·32\times10^{-5}$	0·182	–	–	$1·13\times10^{3}$	$2·34\times10^{-4}$	−0·880	–	$1·22\times10^{2}$	$-8·58\times10^{-1}$	Bluck (1964)
30	Reeds Run	5·2	29·5–86	630–1650	$4·15\times10^{-2}$	$2·64\times10^{-4}$	1·36	−0·955	<0·04	$6·53\times10^{1}$	$1·87\times10^{-4}$	−0·793	3·1	$3·96\times10^{-3}$	$4·08\times10^{-1}$	Brush (1961)
31	Allt Dubhaig	2·3	20–98	458–1809	$2·40\times10^{-2}$	$1·29\times10^{-3}$	2·96	−0·990	<0·04	$7·15\times10^{1}$	$5·97\times10^{-4}$	−0·937	0·3	$4·70\times10^{-5}$	1·15	Ferguson and Ashworth (1961)

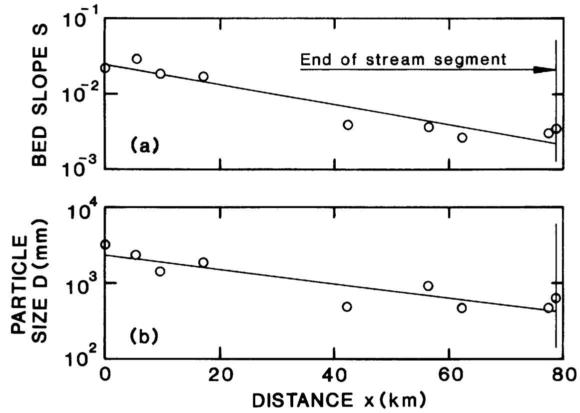

Figure 7. The variation of (a) the bed slope and of (b) the maximum particle size in the Rubicon River, California. Equations 63 and 70 are fitted to the data

to 180 times Bogardi's (1974) average α_1 ($3{\cdot}3 \times 10^{-6}$ m^{-1}), respectively. Gillis Falls, a creek in the Maryland piedmont, is exceptional in having a negative α. This and the Eel River, in the Coast Ranges of northwestern California (Table II), were the only examples found of reverse sorting in natural streams. (Reverse sorting also occurs on deltas formed by combined coarse and fine coal rejects deposited subaerially, but in this case both particle size and specific gravity sorting occur.)

Although most of the short stream segments, including Gillis Falls, are short enough to have exponential profiles even if neither hydraulic sorting nor comminution were dominant (Equation 72), their large α clearly show that sorting is dominant, even if lateral sediment inflow is significant (Equation 55). The low α values of the long stream segments may be partly attributable to relatively large lateral inflows, lower than average comminution rates for fine sediments, or both. Nevertheless, the α data strongly suggest that comminution and hydraulic sorting, respectively, are dominant in long and short stream segments with exponential profiles. The absence of segments of intermediate length (80 km to 240 km) from Table III suggests that neither comminution nor sorting dominates in such segments.

Values of ω were obtained using Equation 75. Although ε varies by a factor of 1000 over the 31 stream segments analysed, ω is limited to the range of 0·082 to 3·77. The equivalent range of dimensionless profiles (Equation 74) is shown in Figure 6. Only the ω for the short segments where sorting is dominant are essentially independent of L.

Values of b were obtained using Equation 69. Values of a for the short stream segments were calculated on the basis that the bed load is exhausted at $x = L$, and the sediment threshold line therefore intersects the low solids concentration threshold lines (Table I) at the D_* corresponding to D_L. The resulting sediment threshold lines (Figure 8) for the sand bed segments of the rivers investigated by Yatsu (1955) lie surprisingly close to the low solids line (Table I); the threshold lines for almost all of the gravel bed segments listed do not, but do generally resemble the threshold lines for gravel bed streams shown in Figure 2. The major exception is Gillis Falls with reverse sorting.

The most complete available data set is that for the Rubicon River (Scott and Gravlee, 1968), which enables a comparison to be made of the threshold line fitted to the flow data (Figure 2) and that based on the sorting data (Figure 8). The two lines match well, although it must be recognized that the two data sets are not completely independent. The estimated standard deviations of a and b based on the flow data are $9{\cdot}8 \times 10^{-4}$ and $1{\cdot}5 \times 10^{-1}$, respectively. The estimates for a and b based on the flow and sorting data differ by less than 70 per cent of these values. Also, an average value of 17·5 m for h', that part of the hydraulic radius associated with surface drag, was calculated using Equations 14 and 15 and the estimates of a, b, S_0 and D_0 based on the sorting data. This value compares well with the range of 7·9 to 29 m for the depth of flow over the river segment (equivalent to the hydraulic radius h) given by Scott and Gravlee (1968).

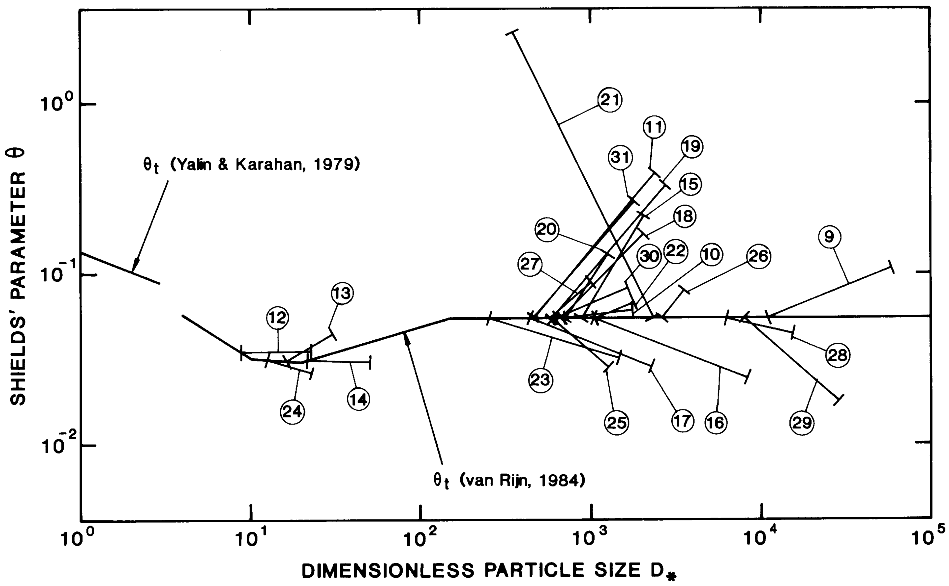

Figure 8. The Shields' θ_t required for entrainment of sediment particles versus the dimensionless particle size D* based on elevation, slope and particle size data and Equations 63, 70 and 73. The lines are identified in Table III

CONCLUSION

A theoretical model of sediment transport and comminution in streams has been presented in which the slope and bed particle size equations for streams with low solids concentrations and low lateral sediment inflows, which have reached equilibrium under steady or quasi-steady hydraulic conditions, are derived. These equations show that exponential longitudinal profiles of any length may occur in such streams, provided that either hydraulic sorting or comminution of the bed load sediments is dominant. Short exponential segments may occur under similar conditions where neither sorting nor comminution is dominant.

Sediment threshold lines and a depth of flow determined on the basis of the theoretical model compare favourably with data derived from field and laboratory studies. It is therefore considered that, although the theoretical model necessarily incorporates numerous approximations and simplications, it nevertheless captures the phenomena essential to the formation of exponential longitudinal profiles in many streams.

A comparison of predictions based on the model, abrasion data, and slope and particle size data from 30 natural stream segments and one alluvial fan with exponential longitudinal profiles strongly suggests that comminution and hydraulic sorting are dominant in long and short natural stream segments, respectively. It also suggests that neither sorting nor comminution is dominant in natural stream segments of intermediate length.

REFERENCES

Adams, J. 1979. 'Wear of unsound pebbles in river headwaters', *Science*, **203**, 171–172.

Andrews, E. D. 1983. 'Entrainment of gravel from naturally sorted riverbed material', *Bulletin of the Geological Society of America,* **94**, 1225–1231.

Ashida, K. and Bayazit, M. 1973. 'Initiation of motion and roughness of flows in steep channels', *Proceedings of the Fifteenth Congress of IAHR,* Istanbul, **1**, 475–484.

Bagnold, R. A. 1956. 'The flow of cohesionless grains in fluids', *Philosophical Transactions of the Royal Society of London, Series A,* **249**, 235–297.

Baker, V. R. 1973. *Paleohydrography and sedimentology of Lake Missoula flooding in eastern Washington'*, Geological Society of America Special Paper **144**, 73 pp.

Bathurst, J. C., Graf, W. H. and Cao, H. H. 1987. 'Bed load discharge equations for steep mountain rivers', in Thorne, C. R., Bathurst, J. C. and Hey, R. D. (Eds), *Sediment Transport in Gravel-bed Rivers,* Wiley, Chichester, 453–477.

Bennett, J. G. 1937. 'Broken coal', *Journal of the Institute of Fuel,* **10**, 22–39.

Blatt, H., Middleton, G. and Murray, R. 1980. *Origin of Sedimentary Rocks,* Prentice-Hall, New Jersey.

Bluck, B. J. 1964. 'Sedimentation of an alluvial fan in southern Nevada', *Journal of Sedimentary Petrology,* **34**, 395–400.

Bogardi, J. 1974. *Sediment Transport in Alluvial Streams*, Akademiai Kiado, Budapest.
Bonnefille, R. 1963. 'Essais de synthese des lois de debut d'entrainment des sediments sous l'action d'un courant en regime uniform', *Bulletin du Centre de Recherche et d'Essais de Chatou,* **5**.
Bradley, W. C., Fahnstock, R. K. and Rowekamp, E. T. 1972. 'Coarse sediment transport by flood flows on Knik River, Alaska', *Bulletin of the Geological Society of America,* **83**, 1261–1284.
Brush, L. M. 1961. *Drainage basins, channels and flow characteristics of selected streams in central Pennsylvania,* US Geological Survey Professional Paper, **282-F**, 145–181.
Carling, P. A. 1983. 'Threshold of coarse sediment transport in broad and narrow natural streams', *Earth Surface Processes and Landforms,* **8**, 1–18.
Carling, P. A. and Reader, N. A. 1982. 'Structure, composition and bulk properties of upland stream gravels', *Earth Surface Processes and Landforms,* **7**, 349–365.
Chen, Y. H., Holly, F. M., Mahmood, K. and Simons, D. B. 1974. 'Transport of material by unsteady flow', in Mahmood, K. and Yevjevich, V. (Eds), *Unsteady Flow in Open Channels,* Water Resources Publications, Fort Collins, Colorado, **1**, 313–365.
Costa, J. E. 1983. 'Paleohydraulic reconstruction of flash-flood peaks from boulder deposits in the Colorado Front Range', *Bulletin of the Geological Society of America,* **94**, 986–1004.
Cunge, J. A., Holly, F. M., Jr and Verwey, A. 1980. *Practical Aspects of Computational River Hydraulics,* Pitman, London.
de Vries, M. 1973. 'River bed variations – aggradation and degradation', *Proceedings of the IAHR International Seminar on Hydraulics of Alluvial Streams,* New Delhi, 1–10.
Everts, C. H. 1973. 'Particle overpassing on flat granular boundaries', *Journal of the Waterways and Habours Division, ASCE,* **99**(WW4), 425–438.
Fahnstock, R. K. 1963. *Morphology and hydrology of a glacial stream – White River, Mount Rainier, Washington,* US Geological Survey Professional Paper, **442-A**, 70 pp.
Ferguson, R. and Ashworth, P. 1991. 'Slope-induced changes in channel character along a gravel-bed stream: the Allt Dubhaig, Scotland', *Earth Surface Processes and Landforms,* **16**, 65–82.
Govers, G. 1987. 'Initiation of motion in overland flow', *Sedimentology,* **24**, 1157–1164.
Graf, W. L. 1988. *Fluvial Processes in Dryland Rivers,* Springer-Verlag, Berlin.
Hack, J. T. 1957. *Studies in longitudinal stream profiles in Virginia and Maryland',* US Geological Survey Professional Paper, **294-B**, 45–97.
Hammond, F. D. C., Heathershaw, A. D. and Langhorne, D. N. 1984. 'A comparison between Shield's threshold criterion and the movement of loosely packed gravel in a tidal channel', *Sedimentology,* **31**, 51–62.
Harder, J. A. and Armacost, I. V. 1966. *Wave propagation in rivers,* University of California, Berkeley, Hydraulic Engineering Laboratory Report **1**, Series 8.
Harrison, S. S. and Clayton, L. 1970. 'Effects of ground-water seepage on fluvial processes', *Bulletin of the Geological Society of America,* **81**, 1217–1226.
Henderson, F. M. 1966. *Open Channel Flow,* Macmillan, New York.
Ibbeken, H. 1983. 'Jointed source rock and fluvial gravels controlled by Rosin's law: a grain-size study in Calabria, south Italy', *Journal of Sedimentary Petrology,* **53**, 1213–1231.
Ippen, A. T. and Verma, P. 1955. 'Motion of particles on bed of a turbulent stream', *Transactions of the ASCE,* **120**, 921–938.
Iwasaki, T. 1967. 'Flood forecasting in the River Kitakami', *Proceedings of the International Hydrology Symposium,* Fort Collins, Colorado, **1**, 103–112.
Kalinski, A. A. 1947. 'Movement of sediment as bed load in rivers', *Transactions of the American Geophysical Union,* **28**, 615–620.
Kamphuis, J. W. 1974. 'Determination of sand roughness for fixed beds', *Journal of Hydraulic Research,* **12**, 193–203.
Kellerhalls, R., Neill, C. R. and Bray, D. I. 1972. *Hydraulic and geomorphic characteristics of rivers in Alberta,* Research Council of Alberta, River Engineering and Hydrology Report **72-1**, 52 pp.
Kittleman, L. R., Jr 1964. 'Application of Rosin's distribution in size-frequency analysis of clastic rocks', *Journal of Sedimentary Petrology,* **34**, 483–502.
Knighton, A. D. 1980. 'Longitudinal changes in size and sorting of stream-bed material in four English rivers', *Bulletin of the Geological Society of America,* **91**, 55–62.
Knighton, A. D. 1982. 'Longitudinal changes in the size and shape of stream bed material: evidence of variable transport conditions', *Catena,* **9**, 25–34.
Komar, P. D. 1987. 'Selective grain entrainment by a current from a bed of mixed sizes: a reanalysis', *Journal of Sedimentary Petrology,* **57**, 203–211.
Komar, P. D. and Li, Z.-L. 1986. 'Pivoting analyses of the select entrainment of sediments by shape and size with application to gravel threshold', *Sedimentology,* **33**, 425–436.
Krumbein, W. D. 1937. 'Sediments and exponential curves', *Journal of Geology,* **45**, 576–601.
Krumbein, W. C. 1938. 'Size frequency distribution of sediments and the normal phi-curve', *Journal of Sedimentary Petrology,* **8**, 84–90.
Martin, C. S. 1970. 'Effect of a porous sand bed on incipient sediment motion', *Water Resources Research,* **6**, 1162–1174.
Millhous, R. T. 1973. *Sediment transport in a gravel-bottomed stream,* PhD thesis, Oregon State University, Corvallis.
Morris, P. H. 1993. 'Two-dimensional model for subaerial deposition of mine tailings slurry', *Transactions of the Institution of Mining and Metallurgy, Series A,* **102**, 181–187.
Morris, P. H. and Williams, D. J. 1996. 'Relative celerities of mobile bed flows with finite solids concentrations', *Journal of the Hydraulics Division, ASCE,* **122**(HY6), 311–315.
Novak, P. and Nalluri, C. 1974. 'Correlation of sediment incipient motion and deposition in pipes and open channels with fixed smooth bed', *Proceedings of the Third International Conference on Hydraulic Transport of Solids in Pipes,* Golden, Colorado, E4/46–56.
Ohmori, H. 1991. 'Change in the mathematical function type describing the longitudinal profile of a river through an evolutionary process', *Journal of Geology,* **99**, 97–110.
Rafay, T. 1964. *Analysis of change of size of bed material along alluvial channels,* MSc thesis, Colorado State University, Fort Collins.
Raudkivi, A. J. 1967. 'Analysis of resistance in alluvial channels', *Journal of the Hydraulics Division, ASCE,* **93**(HY5), 73–84.

Ribberink, J. S. and van der Sande, J. T. M. 1984. *Aggradation in rivers due to overloading,* Delft University of Technology, Department of Civil Engineering, Report no. **84–1**.

Ritter, J. R. 1967. *Bed-material movement, Middle Fork Eel River, California,* US Geological Survey Professional Paper, **575–C**, 219–221.

Schoklitsch, A. 1914. *Uber schleppkraft und geschiebebewegung,* Engelmann, Keipzig.

Schoklitsch, A. 1933. 'Uber die verkleinerung der geschiebe in flusslaufen', *Proceedings of the Vienna Academy of Science, Section IIa,* **142**, 343–366.

Scott, K. M. and Gravlee, G. C. Jr 1968. *Flood surge on the Rubicon River, California: hydrology, hydraulics and boulder transport,* US Geological Survey Professional Paper, **442–M**, 40 pp.

Shields, I. A. 1936. 'Anwendung der Ahnlichkeitsmechanik und der Turbulenzforschung auf die Geschiebebewegung', *Mitteilung der Preussichen Versuchanstalt fur Wasserbau und Schiffbau,* **26**.

Shih, S.-M. and Komar, P. D. 1990. 'Hydraulic control of grain size distributions of bed load gravels in Oak Creek, Oregon', *Sedimentology,* **37**, 367–376.

Shulits, S. 1941. 'Rational equation of river-bed profiles', *Transactions of the American Geophysical Union,* **22**, 622–630.

Simons, D. B. 1977. 'River and canal morphology', in Shen, H. W. (Ed.), *River Mechanics, Vol II,* H. W. Shen, Fort Collins, 20·1–57.

Sokolnikoff, I. S. and Redheffer, R. M. 1958. *Mathematics of Physics and Modern Engineering,* McGraw-Hill, New York.

Sternberg, H. 1875. 'Intersuchunge uber des langen- und querprofil geschiebefuhrender flusse' *Zeitschrift fur Bauwesen,* **25**, 483–506.

Tanner, W. F. 1971. 'The river profile', *Journal of Geology,* **79**, 482–492.

Tarantola, A. 1987. *Inverse Problem Theory,* Elsevier, Amsterdam.

Thomas, G. B. 1968. *Calculus and Analytical Geometry,* Addison-Wesley, Reading, Massachusetts.

van Rijn, L. C. 1984. 'Sediment transport, part I: bed load transport', *Journal of the Hydraulics Division, ASCE,* **110**, 1431–1456.

Williams, D. J. and Morris, P. H. 1989. 'Comparison of two models for subaerial deposition of mine tailings slurry', *Transactions of the Institution of Mining and Metallurgy, Series A,* **98**, 73–77.

Yalin, M. S. 1992. *River Mechanics,* Pergamon, Oxford.

Yalin, M. S. and Karahan, E. 1979. 'Inception of sediment transport', *Journal of the Hydraulics Division, ASCE,* **105**, 1433–1443.

Yatsu, E. 1955. 'On the longitudinal profile of the graded river', *Transactions of the American Geophysical Union,* **36**, 655–663.

Yu, Y. S. and McNown, J. S. 1964. 'Runoff from impervious surfaces', *Journal of Hydraulic Research,* **2**, 3–24.